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ON THE FOUNDATIONS OF A THEORY OF EQUILIBRIUM CRACKS IN ELASTIC SOLIDS

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Propositions expressed in [9] receive further development herein, are refined and systematized. Proceeding from the law of interaction between atoms, a conception is proposed which considers cracks in elastic solids as nontrivial modes of equilibrium deformation. Crack formation is treated as the loss of stability (in the large) of trivial equilibrium modes. The formulation of the brittle fracture criterion in the neighborhood of the end of the crack is refined. The carrying capacity of a solid having an equilibrium crack is estimated approximately in an example of the A. Griffith problem.

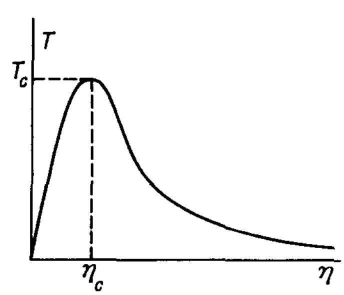

Fig. 1

1. Let $T = D\sigma$ (where $D = 2a$ is the atomic diameter) be the force of interaction between two parallel rows of atoms, referred to unit length of these rows. The dependence of T on the change in spacing between the atom centers 2η is as shown in Fig. 1. One of the suitable approximations of this dependence will be

$$T = D\sigma = 2E\eta e^{-\eta/\eta_c} \tag{1.1}$$

where η_c is the value of η corresponding to the maximum value of T ($T_c = D\sigma_c$, E is Young's modulus, σ_c is the rupture yield strength). The relationships

$$T_m = 2E\eta_c e^{-1}, \qquad \sigma_c = \frac{E\eta_c}{a} e^{-1} \tag{1.2}$$

$$\sigma = \sigma_c \frac{\eta}{\eta_c} e^{1-\eta/\eta_c} \tag{1.3}$$

result from (1.1), on whose basis

$$\gamma = \frac{1}{2}\int_{\eta_c}^{\infty} \sigma d\eta = \eta_c\sigma_c = e\frac{\sigma_c^2 a}{E} \approx 1.35\frac{D\sigma_c^2}{E} \tag{1.4}$$

where γ is the density of the surface energy of the elastic solid. The result (1.4) differs from the following known formula

$$\gamma = \frac{1}{2}\frac{\sigma_c^2 D}{E} \tag{1.5}$$

by a numerical coefficient. However, (1.5), having been deduced under the assumption that the curve $\sigma \sim \eta$ is approximated by half a sinusoid [1], will probably lower γ. It is understood that other kinds of dependences of σ on η can also be considered, for example

$$\sigma = \frac{E}{k}\frac{1}{(1+\eta/a)^m}\left[1 - \frac{1}{(1+\eta/a)^k}\right] \tag{1.6}$$

The relationships

$$\sigma_c = \frac{E}{m}\frac{1}{(1+k/m)^{(1+m)/k}}, \qquad \eta_c = D\left[\left(1+\frac{k}{m}\right)^{1/k} - 1\right]$$

$$\gamma = ED\,\frac{2m+k-1}{2m(m-1)(m+k-1)}\,\frac{1}{(1+k/m)^{(m+k-1)/k}} \tag{1.7}$$

correspond to this expression, and are more complicated than (1.2), (1.4); this obliges giving (1.1) preference over (1.6). The qualitative nature of the subsequent investigation does not permit detailed examination of the question which of the two approximations presented affords the possibility of a closer approach to reality.

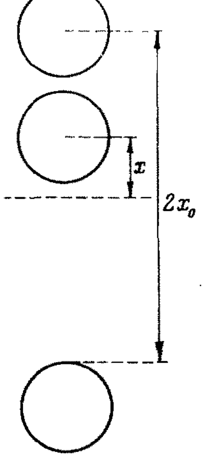

Fig. 2

2. Let us examine the following situation (Fig. 2). Let the spacing between two fixed atoms in space be $2x_0 \geqslant D + 4\eta_c$ and let there be an atom between them which is subject only to interaction forces from the fixed atoms. Assuming these forces to be subject to the law (1.1), it is easy to see that the intermediate atom has two equilibrium positions $x_{2,3} = \pm x^*$, as well as the trivial equilibrium position $x_1 = 0$, where x_2, x_3 are roots of the transcendental equation

$$x = (x_0 - {}^1/_2D)\ \mathrm{th}\ ({}^1/_2 x\eta_c) \tag{2.1}$$

The trivial equilibrium position is unstable, and the two other equilibrium positions are stable. Hence, the outer atom, turning out to be in interaction with two adjacent atoms according to the law of the descending portion of the $\sigma \sim \eta$ curve, will inevitably be attracted to one of them, and thereby interact with it according to the law of the ascending portion of the $\sigma \sim \eta$ curve.

It is hence seen that equilibrium of atoms interacting according to the law of the descending portion of the $\sigma \sim \eta$ curve, is unstable. If all the atoms of the solid turned out to be in such a state, then it would have lost its capacity to resist deformation. Moreover, it could not resist deformation even if all the atoms of the two adjacent layers intersecting the solid turned out to be in unstable interaction. From the above, the deduction follows that interaction according to the law of the descending portion of the $\sigma \sim \eta$ curve can only exist locally in an elastic solid in a state of stable equilibrium deformation. It can originate only on portions of adjacent atomic layers under the condition that atoms which do not belong to these portions are in a state of stable interaction with atoms of the other atomic layers. As will be clarified later, the greatest spacings between atomic layers in the equilibrium of elastic solids within which are portions of atomic layers interacting according to the law of the descending branch of the $\sigma \sim \eta$ curve will, as a rule, essentially exceed the atomic diameter within the limits of such portions. Consequently, such portions can be spoken of as slots spoiling the continuity of the solid. All the atoms around these slots are in a state of stable interaction according to the law of the ascending branch of the $\sigma \sim \eta$ curve, whereupon the spacing between atoms in the neighborhood of the slots is $x < D + 2\eta_c$, and it can be considered that the continuity of the solid is conserved here.

On the basis of the above, in a theoretical investigation of equilibrium deformations of elastic solids, the solid can always be treated as a continuum by using methods of elasticity theory. If desired, however, not only the equilibrium modes when all the atoms interact according to the law of the ascending (stable) branch of the $\sigma \sim \eta$ curve can be taken into account, but also the modes when slotlike domains unfilled with atoms between whose edges there is interaction according to the law of the descending branch of the $\sigma \sim \eta$ curve, originate in the solid. It is then necessary to admit the possibility of the appearance of the new boundaries in the shape of cuts within the solid. The shape and size of these cuts are not known in advance. They can be determined from elasticity theory equations for appropriate boundary conditions, resulting from (1.1) for $\eta > \eta_c$, being given on the edges of each slot.

The strict formulation and solution of the described, essentially nonlinear, problem are quite difficult, whereupon it is expedient to suggest an approximate approach in the first stage of the investigation, which would entrain at least the qualitative aspect of the phenomenon being studied. Such an approach can be formulated on the basis of the following simplifying assumptions:

1) the relationship between the stresses and strains on the ascending (stable) portion of the curve $(\eta < \eta_c)$, i.e. in the whole domain of the solid where its continuity is conserved, is approximated by Hooke's law;

2) the equilibrium equations and formulas connecting the strains and displacements are taken in linear form, i.e. the problem is treated as being geometrically linear;

3) the descending portion of the $\sigma \sim \eta$ dependence is approximated by the step curve

$$\sigma = \sum_{k=1}^{n} \sigma_k \delta_0 (\eta_k - \eta_*) \qquad (\eta_* = \eta - \eta_c) \tag{2.2}$$

Here $\delta_0(x)$ is the Heaviside function

$$\delta_0 (x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{2.3}$$

The constants σ_k, η_k can be selected from considerations of the best approximation of (2.2) to the $\sigma \sim \eta$ curve for $\eta \geqslant \eta_c$. Acceptance of the listed simplifications is equivalent to linearization of all the equations of the problem being considered, and permits approximate solutions to be obtained for specific problems.

3. The simplest imaginable modification described by the approximate theory is obtained if just the first term is kept in (2.2) by putting

$$\sigma = \sigma_1 \delta_0 (\eta_1 - \eta_*) \qquad (\eta_* > 0) \tag{3.1}$$

It should hence be assumed that $\sigma_1 = \sigma_c$, and η_1 should be determined from the condition

$$\sigma_1 \eta_1 = \sigma_c \eta_1 = \gamma \tag{3.2}$$

thereby demanding that the area of the approximating curve (Fig. 3) be equal to the area of the curve (1 1) in the range $\eta_c \leqslant \eta \leqslant \infty$. This condition is equivalent to the requirement that the approximating dependence yield the same value of the surface energy density as (1.1). It recalls the linearization principle widely utilized in the theory of nonlinear oscillations, which is based on equating the work of nonlinear forces to the work of the approximately linear forces replacing them within the argument range characteristic for the problem.

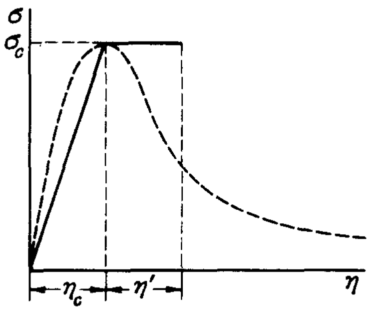

Fig. 3

In this most simple formulation, let us consider the problem of the equilibrium of an elastic isotropic plane subjected to tensile stresses given at infinite $\sigma_{yy}(x, \pm\infty) = \sigma$.

This problem has the trivial solution

$$\sigma_{xx}(x, y) = \sigma_{xy}(x, y) = 0, \qquad \sigma_{yy}(x, y) = \sigma \tag{3.3}$$

However, this solution is not unique if the nonlinearity of the connection between the stresses and strains according to (1.1) is taken into account.

Without posing the problem of seeking all nontrivial equilibrium modes of deformation of the solid which are possible in this case, let us limit ourselves to an investigation of the possibility of the existence of modes for which an infinitely long plane slot perpendicular to the tension direction, will be formed at an arbitrary point of the solid, i.e. let us examine the case investigated by Griffiths [2].

It hence turns out to be possible to utilize the already known solution of Leonov and Panasiuk [3, 4], who came up with the useful idea of approximating the descending portions of the $\sigma \sim \eta$ curve by formulas (3.1), (3.2).

According to the authors cited

$$\sigma_{yy}(x, 0) = \sigma_c + \frac{x}{\sqrt{x^2 - l^2}}\left[\sigma - \frac{2\sigma_c}{\pi}\arccos\frac{l_0}{l}\right] + \\ + \frac{\sigma_c}{\pi}\left[\arcsin\frac{l^2 - xl_0}{l(x - l_0)} - \arcsin\frac{l^2 + xl_0}{l(x + l_0)}\right] \tag{3.4}$$

$$v(x, 0) = \eta^*(x, 0) = \frac{2}{E}\left[\sigma - \frac{2}{\pi}\sigma_c \arccos\frac{l_0}{l}\right]\sqrt{l^2 - x^2} + \\ + \frac{\sigma_c}{\pi E}\left[(x - l_0)\Gamma(l, x, l_0) - (x + l_0)\Gamma(l, x, -l_0)\right] \tag{3.5}$$

$$(|x| \leqslant l)$$

Here $2l = L$ is the slot length, $l - l_0 = \Delta$ is the width of the portion of the slot (at its endpoint), within whose limits there is interaction according to the law (3.1) between the edges

$$\Gamma(l, x, k) = \ln \frac{l^2 - xk - \sqrt{(l^2 - x^2)(l^2 - k^2)}}{l^2 - xk + \sqrt{(l^2 - x^2)(l^2 - k^2)}} \tag{3.6}$$

The y-axis is perpendicular to the slot, and the x-axis is along the slot from the origin to the midpoint.

Formula (3.4) yields the distribution of the normal stress σ along the x-axis (for $|x| \geqslant l$), and (3.5) defines the equilibrium mode of the twisting of the slot "edges".

The quantities given in these formulas will be σ, σ_c, E. As regards the length of the slot and the width of the interaction portions of the slot edges Δ they cannot be taken arbitrarily. They are determined uniquely if the relationships (3.1), (3.2) are taken into account, and which have not been utilized in (3.4), (3.5). But before forming the appropriate equations, it is expedient to simplify the expressions (3.4)–(3.6), which is possible if it is taken into account that only cases when

$$\frac{\sigma}{\sigma_c} \ll 1, \qquad \frac{\Delta}{l} \ll 1 \tag{3.7}$$

are of practical interest.

The first inequality results from the fact that the ultimate tensile strength σ_c of a lattice without defects is on the order of the Young's modulus, as is seen at least from (1.2.2). The validity of the second simplification is verified by subsequent computations according to which Δ turns out to be a quantity on the order of the atomic diameter D. Because of (3.7), the expressions (3.4), (3.5) can be replaced by the following asymptotic formulas (which are valid in the neighborhood of the slot endpoints):

$$\sigma_y(\xi, 0) = \frac{1}{2}\sigma_c + \frac{1}{\sqrt{2}}\sqrt{\frac{l}{\xi}}\left[\sigma - \frac{2\sqrt{2}\sigma_c}{\pi}\sqrt{\frac{\Delta}{l}}\right] + \frac{\sigma_c}{\pi}\arcsin\left(\frac{\Delta - \xi}{\Delta + \xi}\right) \tag{3.8}$$

$$v(\xi^*, 0) = \frac{2\sqrt{2}}{E}\left[\sigma - \frac{2\sqrt{2}}{\pi}\sigma_c\sqrt{\frac{\Delta}{l}}\right]\sqrt{l\xi^*} + \frac{4\sigma_c}{\pi E}\sqrt{\Delta\xi^*} + \tag{3.9}$$

$$+ \frac{\sigma_c}{\pi E}(\Delta - \xi)\ln\frac{(\sqrt{\Delta} - \sqrt{\xi})^2}{(\sqrt{\Delta} + \sqrt{\xi})^2} \qquad (x = l + \xi,\ x = l - \xi^*)$$

Quantities on the order of Δ/l, ξ / l, ξ^* / l, as well as products of these small quantities, were discarded successively in comparison to one in the derivation of (3.8), (3.9) from (3.4)–(3.6).

It follows from (3.8) that for

$$\sigma - \frac{2\sqrt{2}\sigma_c}{\pi}\sqrt{\frac{\Delta}{l}} > 0 \tag{3.10}$$

infinitely large tensile stresses will originate at the slot ends, and for

$$\sigma - \frac{2\sqrt{2}\sigma_c}{\pi}\sqrt{\frac{\Delta}{l}} < 0 \tag{3.11}$$

infinitely large compressive stresses. If it is assumed that neither are admissible, then we arrive at the condition

$$\sigma - \frac{2\sqrt{2}\sigma_c}{\pi}\sqrt{\frac{\Delta}{l}} = 0 \tag{3.12}$$

which is a formulation, for the specific problem under consideration, of the Khristianovich-Barenblatt postulate [5, 6], according to which solutions with infinite stresses in the neighborhood of the endpoints should be considered physically incorrect in problems of the deformation of solids having crack-cuts, and eliminated from consideration.

Somewhat later this postulate will be discussed and rejected. However, it is first expedient to agree with (3. 12) and to examine the consequences resulting from it. In addition to (3. 12), the parameters of the problem should be subject also to the relationship

$$v(\Delta, 0) = \frac{4\sigma_c \Delta}{\pi E} \frac{\gamma}{\sigma_c} \tag{3.13}$$

which follows from (3. 9), and formulas (3. 1), (3. 2) taking into account that $\eta_1 = \gamma / \sigma_c$.

Eliminating Δ from (3, 12) and (3. 13), we arrive at the equality

$$L = 2l = L_g = \frac{4}{\pi} \frac{E\gamma}{\sigma^2} \tag{3.14}$$

which is identical to the known Griffith formula.

Therefore, if the postulate of finiteness of the stresses (both tensile and compressive) is accepted, then it turns out that a completely definite length of an equilibrium slot will correspond to each value of σ. The slot may not be greater than this length since infinite tensile stresses could hence occur in the neighborhood of its endpoints, and cannot be less than this length since then infinite compressive stresses would originate in the neighborhood of its ends.

It should be noted that the authors of the considered solution [3, 4] consider its application possible even to cracks of length $L < L_g$. However they must here assume that there is no interaction between the slit edges in the portion $|x| \leqslant l_0$ even if

$$v(l_0, 0) < \gamma / \sigma_c \tag{3.15}$$

The fact is that the cited authors treat a slot as a real, previously assigned infinitely thin cut in the solid. But the question then arises how can the slot edges diverge in the presence of atomic interaction forces? In order to overcome this difficulty, the existence of a seemingly infinitely thin shield, impervious to the interatomic forces, is sometimes postulated tacitly and sometimes explicitly in the plane of the cut in the theory of crack-cuts.

Taken as a length in the theory [3, 4] is $2l_0$, and the crack is considered to be "shielded" from interaction between edges along this whole length, independently of the spacing between them. Hence, condition (3. 13) is written as

$$v(\Delta, 0) \leqslant \gamma / \sigma_c \tag{3.16}$$

where the critical length of an equilibrium crack $L = L_g$ corresponds to the equality sign.

The treatment proposed above of the crack-cuts as nontrivial equilibrium modes of the elastic solids with physically nonlinear properties permits getting rid of the "mythical" shield which is impervious to atomic interaction forces.

The origination of cracks in such a treatment is considered as the loss of stability (in the large) of the trivial mode of body deformation, i. e. as an effect analogous, say, to the snapping phenomenon in shells. In such an approach, the presence of some cut in the solid is not required for slot formation, and the need for shielding the interatomic forces drops out. The transition of the solid into an equilibrium state with a slot is however connected with overcoming the energy barrier proportional to the slot area. In real solids, there is hence apparently sufficient local circumstances contributing to the realization of the transition from the trivial to the nontrivial equilibrium mode, exactly as there are always sufficient reasons in real shells to assure overcoming the energy barrier standing in the path of snapping dent formation.

4. Therefore, according to the approximate solution presented above, the length of an equilibrium crack of the kind considered is completely definite: $L = L_g$. The slot hence always turns out to be unstable since an arbitrarily small finite change in the external loading on the solid $\sigma \pm \Delta\sigma$ results in the origination of infinite stresses, forbidden by the Khristianovich-Barenblatt principle, in the neighborhood of the ends of the slot. The certainty of this principle, at least with respect to tensile stresses, is indubitable at first glance since the law of the stress-strain relation (1. 1) constrains the former to the finite quantity σ_c. And even the possibility of real solids supporting infinite compressive stresses seems to be completely improbable.

However, let us show that the solution of elasticity theory problems with infinitely large stresses should by far not always be rejected. The fact is that the rupture of solids is a discrete process; it is impossible to separate half an atom from half of another, say, while retaining the connection between their two remaining halves. Failure of the connection in a pair of atoms, i. e. achievement of the limit value of the cohesion force T_m in some pair of atoms, will be a rupture "quantum". Since $T = D\sigma$, the inequality

$$\sigma_n \geqslant \sigma_c \tag{4.1}$$

will be the failure condition, where σ_n is the maximum value of the normal tensile stress. However, cases are possible when the gradient of the normal stress in the neighborhood of its greatest value is so large that it is not possible to neglect the change in σ_n even within the limits of one atomic diameter.

Then

$$T = \iint \sigma_n d\Omega \tag{4.2}$$

where integration is within a square with side l_0. (Here and henceforth, keeping in mind the roughness of the subsequent considerations which rely on the apparatus of linear elasticity theory, it is meaningless to devote oneself to the peculiarities in the construction of atomic lattices. These latter are treated as simple, cubic, loosely packed lattices.)

In the above-mentioned special cases (4. 1) should be replaced by the inequality

$$\frac{1}{D^2}\iint \sigma_n d\Omega \geqslant \sigma_c \tag{4.3}$$

As soon as the reverse inequality

$$\frac{1}{D^2}\iint \sigma_n d\Omega < \sigma_c \tag{4.4}$$

is satisfied at all points of the solid, the strength of the latter is known to be assured since the external forces acting on the solid are now inadequate for the maximum cohesion to be overcome by at least one pair of atoms.

However, it does not follow from this that condition (4. 3) can be considered the fracture criterion. It will be the condition for failure of just one element of the quite complex multiply statically indeterminate system of the atomic lattice. Hence, the inequality (4. 3), being the necessary fracture criterion, will not generally be sufficient. The formulation of a sufficient criterion for brittle fracture should be connected with an estimation of the carrying capacity of the atomic lattice of the solid subjected to a given external loading.

In practice, the discrete criterion (4. 3) differs from the continual criterion (4. 1) only in the neighborhoods of singular points of the stress field, where the infiniteness of the stress, absolutely forbidden by the criterion (4. 1), does not, by far, always turn out to contradict the criterion (4. 4), which assures the strength of the atomic lattice. Let us illustrate this by a most simple example.

5. Let us consider the problem of tension on a plane with a cut by stresses $\sigma_y\ (x, \pm \pm \infty) = \sigma$ by neglecting interaction between the cut edges here (the Griffith problem). We consider the x-axis to coincide with the cut, and take the origin at one of the end-points. The normal stress distribution σ_y near the end of the crack is determined by the known asymptotic formula of Sneddon [7]

$$\sigma_y = \sigma\left(1 + \frac{1}{2}\sqrt{\frac{L}{x}}\right) \tag{5.1}$$

There results from (5. 1) that $\sigma_y \to \infty$ as $x \to 0$, and therefore, the strength condition $\sigma_n \leqslant \sigma_c$ at the ends of the crack is not satisfied for any crack length L. For the particular case under consideration, the discrete criterion (4. 4) becomes

$$\frac{1}{D}\int_0^D \sigma_y dx = \sigma\left(1 + \sqrt{\frac{L}{D}}\right) < \sigma_c \tag{5.2}$$

Hence, the strength of the solid is known to be assured if

$$L < D\,\frac{(\sigma_c - \sigma)^2}{\sigma^2} \approx \frac{D\sigma_c^2}{\sigma^2} \tag{5.3}$$

By utilizing (1. 2), this expression can be reduced to

$$L \leqslant 0.74\,\frac{E\gamma}{\sigma^2} = 0.58 L_g \tag{5.4}$$

where L_g is the critical crack length according to Griffith. If not (1. 2) but the alternative formula of Orowan (1. 5) is used to transform (5. 3), we will then have

$$L \leqslant \frac{2E\gamma}{\sigma^2} = 1.56 L_g \tag{5.5}$$

Therefore, (1. 2) and (1. 5), corresponding to two distinct methods of approximating the atomic interaction curve utilized in transforming (5.3), yield a somewhat understated and a somewhat exaggerated value of the critical crack length as compared to the result (3. 14) of A. Griffith.

A theory of equilibrium cracks, which dispenses with the postulate of finiteness of the stresses, will be presented in Sect. 7, in which the considered example will appear as a particular case. Then the interrelations between (5. 3) and the Griffith formula (3. 14) will become more explicit, where it turns out that the analysis presented above is not rigorous. However, it should be retained in research as one of the steps in an investigation which demonstrates most simply that the discrete strength criterion (4. 4), applied to the Griffith problem, will yield quantitative results similar to those which the mentioned author obtained from energy considerations.

The following deductions result from the considered illustration.

a) At points of the solid where the stresses and their gradients are infinite, the stresses averaged within the limits of a single atom should be considered the strength characteristic

$$\bar{\sigma}_n = \frac{1}{D^2}\int \sigma_n d\Omega \tag{5.6}$$

b) The strength criterion (4. 4) when applied to estimate crack stability will yield results agreeing with the Griffith energy theory not only qualitatively but also quantitatively.

c) The existence of singular points in the stress field still does not mean physical incorrectness of the solution. Upon compliance with (4.4), such solutions are admissible. Moreover, since the condition (4.4) is only necessary but not sufficient, it will generally understate the domain of loadings which will not spoil the strength of the solid. A number of authors turned attention to the need for the transition from stresses to interatomic forces in estimating the strength of a solid at the ends of a crack. The founder of the theory of brittle fracture, Griffith [2], understood this well. Elliot [8] examined a discrete strength condition at a crack end in a form different from that elucidated.

6. In connection with the discussions of the two preceeding sections, a doubt can arise as to whether it is admissible to consider spacings on the order of an atomic diameter as finite quantities in solutions of problems obtained from the elasticity theory equations, and can one count upon obtaining at least qualitatively correct results.

However, the following reasoning favors this possibility. Real solids which have a discrete configuration are approximated in continuum mechanics by bodies consisting of infinitesimal particles. The stress-strain relationships in a continuous solid are hence taken such that when they are averaged within the dimensions of one atom relationships are obtained which express the interaction between two atoms. Requiring a rather common, but then descriptive expression, it can be said that the discrete interaction between atoms seems "to be spread" over the whole volume of the solid. However, this does not mean neglecting the size of the atoms. Let us clarify this last sentence by an illustration taken from shell theory. In this theory a method is often used to analyze stiffener-reinforced shells which is based on the distribution of the rib stiffness along the intervening span. The ribbed shell is hence converted into an anisotropic smooth shell of approximately equivalent mechanical properties. The discrete finite-stiffness ribs are hence replaced by infinitely close ribs of infinitesimal width. It is quite undestandable that this is just a computational method. Its utilization does not mean at all that the size of the actual discrete stiffeners or the spacing between them can be neglected in a practical application of the results emerging therefrom. Thus, for example, when desiring to determine the bending stresses in the stiffeners, the bending moment should be obtained by taking values of the change in curvature along the considered stiffener from an analysis of the fictitious structurally anisotropic shell, and multiplying the curvature by EI, where I is the moment of inertia of the stiffener.

It is seen from the illustration given, that methods based on approximate replacement of discrete by continuous mechanical systems do not exclude the possibility of obtaining information on the stress resultants originating in the elements of the discrete systems. Were it otherwise, such methods would be deprived of practical value. The appearance of infinite stresses in the solution of some elasticity theory problems indicates that the interatomic forces in the atomic lattice in such problems will change rapidly with the passage from one pair of atoms to another. Infinite stresses and their gradients will naturally be obtained at appropriate points when a real discrete lattice is replaced by a continuum (i.e. lettig D tend to zero). However, when converting stresses into interatomic forces, the latter turn out to be finite, as is seen from the example presented in the preceding section. The satisfactory quntitative agreement between the results obtained by this means and the Griffith energy theory can be considered as confirmation of the admissibility of the approach described for the estimation of the strength of brittle solids.

7. Let us now return to Sect. 3, wherein (3.14) was derived which uniquely defines the length of an equilibrium crack. This formula has been obtained from equalities (3. 8), (3. 9) upon which conditions (3. 12) and (3. 13) have been imposed. The second results from the approximation taken for the law of interatomic coupling on the descending branch of the $\sigma \sim \eta$ curve (3. 1), (3. 2), and the first expresses the requirement of finiteness of the stresses at the ends of the crack. This latter (as has been shown above) is not absolute, and should be replaced by (4. 4), which becomes in the plane problem being considered

$$\frac{1}{D}\int_0^D \sigma_y dx \leqslant \sigma_c \tag{7.1}$$

The critical state of a crack for which the interaction forces between pairs of atoms closest to its endpoints reach the limit value T_m corresponds to the equality sign here. Let us determine the critical crack length $L_k = 2l_k$ corresponding to this state. Substituting (3. 8) into (7. 1), and using this formula with the equality sign, we obtain

$$2\sqrt{\frac{L_k}{D}}\frac{\sigma}{\sigma_c} = \frac{4}{\pi}\sqrt{\alpha} + (1+\alpha)\left[1 - \frac{2}{\pi}\arcsin\left(\frac{\alpha-1}{\alpha+1}\right)\right] \qquad \left(\alpha = \frac{\Delta}{D}\right) \tag{7.2}$$

Taking account of the arbitrariness of condition (3. 11), Eq (3. 13) will be

$$v(\Delta, 0) = \frac{2}{E}\left[\sigma\sqrt{L_k\Delta} - \frac{2}{\pi}\sigma_c\Delta\right] = \frac{\gamma}{\sigma_c} \tag{7.3}$$

If the following notation is introduced

$$L_g = 2l_g = \frac{4}{\pi}\frac{E\gamma}{\sigma^2}, \qquad \beta = \frac{\gamma E}{D\sigma_c^2}, \qquad U^2 = \frac{\pi}{4}\frac{\beta}{\alpha} \tag{7.4}$$

(7. 3) can be written either as

$$2\frac{\sigma}{\sigma_c}\sqrt{\frac{L_k}{D}} = \frac{\beta}{\sqrt{\alpha}} + \frac{4}{\pi}\sqrt{\alpha} \tag{7.5}$$

or as

$$2\sqrt{\frac{L_k}{L_g}} = U + \frac{1}{U} \tag{7.6}$$

It is hence seen that $L_k > L_g$. Furthermore, subtracting (7. 5) from (7. 2), we find

$$\beta = \sqrt{\alpha}(1+\alpha)\left[1 - \frac{2}{\pi}\arcsin\frac{\alpha-1}{\alpha+1}\right] \tag{7.7}$$

Here β is a function of only the physical constants of the problem, i. e. is given. Hence, (7. 7) will be a transcendental equation in the single unknown α.

A graph illustrating the dependence of $\sqrt[4]{\alpha}$ on β is presented in Fig. 4. On the basis of this graph the coefficients λ and $\varkappa$ in the formulas

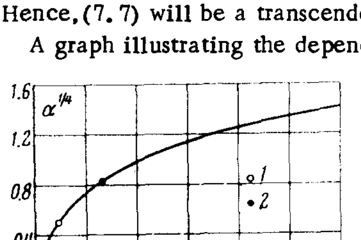

Fig. 4

$$L_k = \lambda\frac{D\sigma_c^2}{\sigma^2}, \qquad L_g = \varkappa L_k \tag{7.8}$$

$$\lambda = \frac{2}{\pi}\beta + \frac{4}{\pi^2}\alpha + \frac{1}{4}\frac{\beta^2}{\alpha} \tag{7.9}$$

$$\varkappa = \frac{4U^2}{(1+U^2)^2} \tag{7.10}$$

can be calculated by using (7. 4. 2), (7. 5), (7. 6).

These expressions which connect λ and $\varkappa$ to α, β, U, result from (7. 5), (7. 6). Curves of the dependences of λ and $\varkappa$ on β are presented in Figs. 5 and 6. It is interesting to note the two limit cases $\beta \to 0$ and $\beta \to \infty$. In the former $\gamma = 0$, which corresponds to neglecting interatomic forces on the descending portion of the $\sigma \sim \eta$ curve. Hence $L_g = 0$. However, L_k turns out to be finite because finiteness of the atom size has been taken into account. The solution corresponding to this case has already been obtained in Sect. 5.

The other extreme case ($\beta \to \infty$) corresponds to the assumption that $D = 0$, i.e. the passage to a continuum with a finite value of γ. Then it is natural that $L_k = L_g$ since precisely such a formulation of the problem will correspond to conditions for the derivation of the Griffith formula. It follows from the solution constructed that L_k does not generally agree with L_g, where $L_k \geqslant L_g$ always. But L_k will be the upper bound for the length L of the equilibrium slot. The question arises as to what is the lower bound for this quantity

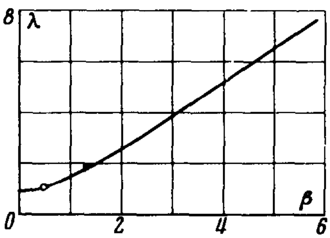

Fig. 5

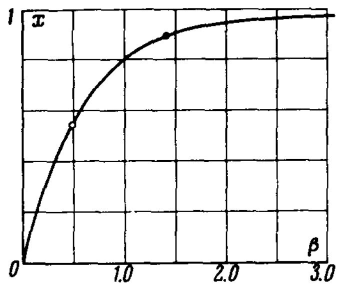

Fig. 6

To answer this, let us condider how the equation governing the slot size will change if

$$\frac{1}{D}\int_0^D \sigma_y dx < \sigma_c$$

Hence, (7.2) is converted into the inequality

$$2\sqrt{\frac{L}{D}}\frac{\sigma}{\sigma_c} < \frac{4}{\pi}\sqrt{\alpha} + (1+\alpha)\left[1 - \frac{2}{\pi}\arcsin\left(\frac{\alpha-1}{\alpha+1}\right)\right] \qquad (7.11)$$

and (7.3) remains unchanged. Correspondingly, all the formulas (7.5), (7.6) are retained since they will be consequences of (7.2) only. Hence, for any (and not only for the critical crack length)

$$2\sqrt{\frac{L}{L_g}} = U + \frac{1}{U} \qquad (7.12)$$

It hence follows that $L \geqslant L_g$. It therefore turns out that the possible equilibrium crack lengths are bounded by the range

$$L_g \leqslant L \leqslant L_k \qquad (7.13)$$

For $\beta = 0$ $(\gamma = 0)$ this range is transformed into $0 \leqslant L \leqslant L_k$, i.e. for solids with negligible surface energy, equilibrium cracks of any length less than the critical are possible.

For $\beta \to \infty$ $(D = 0)$ this interval shrinks to a point

$$L = L_k = L_g$$

i.e. for a continuum (a medium with infinitesimal atoms) the origination of equilibrium cracks of just one completely specific length is possible. As is seen from the graph (Fig. 6), L_g varies between $0.6L_k \leqslant L_g < 0.9\,L_k$ in the most probable range of variation of β $(0.5 < \beta < 1.3)$ corresponding to (1.4), (1.3).

The result (7.13) is rather unexpected. It follows therefrom that although equilibrium cracks with infinite stresses at the ends can indeed exist, it is however required by all means that these stresses be positive. Equilibrium cracks with negative infinite stresses do not exist. However, the possibility is not excluded that this result is not final, as a more accurate analysis will extend the interval (7.13) somewhat on the left side.

The fact is that according to the solution elucidated, it turns out that for $L < L_g$ the crack edges begin to intersect near its endpoints, which is physically absurd. This is precisely why this case is indeed forbidden by the constructed solution. However, the noted absurdity is inherent not in the problem under consideration, but only in the idealized formulation in which it has been examined.

Indeed, according to the conception expounded, the crack is a cavity between two rows of atoms within whose limits the spacing between the atoms exceeds $D + 2\eta_c$. The displacements $v(x, 0)$ governing the shape of this cavity were defined as

$$v(x, 0) = \eta - \eta_c = \eta^*$$

But in such a definition negative values of $v(x, 0)$ are not excluded at all, and do not contradict the physical meaning.

Their appearance means only that the atomic rows approach to a distance $\eta < D + 2\eta_c$ on appropriate portions of the whole length, i.e. that the atoms go from interaction according to the law of the descending portion of the $\sigma \sim \eta$ curve over to interaction according to the law of the ascending portion of the curve.

The atomic diameter in the solution presented above, which is considered finite in the range $|x| < 0.5L$ (since a curve of the atomic interaction in the form (1.1) is used here which connects the stresses with the remaining atom displacements), is taken equal to zero in the interval $|x| > 0.5L$ (since the continuum model is used here). This latter indeed results in the fact that the physically admissible case is converted into an absurdity. Some doubt hence arises about the lower bound of the inequality (7.13).

As regards its upper bound, some doubt can also be expressed here on the basis of the fact that the fracture criterion (4.3), which was utilized to define this boundary, will only be necessary but not generally sufficient. However, in the problem under consideration this criterion is apparently not only necessary but also sufficient.

Indeed, if the interatomic force for two pairs of atoms closest to the slot endpoints reaches the limit value T_m, then this means substantially that the slot length is magnified by $2D$. But then the next two pairs of atoms turn out to be in the same situation as were the two preceding pairs of atoms, and therefore, the interatomic forces for them should reach the limit value T_m. A slot for which $L > L_k$ will thereby be broadened, although fracture of the solid will not yet occur. It should hence be considered that the value $L = L_k$ corresponds to exhaustion of the carrying capacity of a body having a slot, i.e. the upper bound of (7.13) is not needed in a correction in principle.

It is understood that the quantitative results emerging from the solution presented should be considered only as a first approximation. It is desireable to refine them for a more accurate investigation of the situation in the neighborhood of the slot ends.

8. According to the theory elucidated, some interval of values of L corresponds to each value of the stress σ at infinity, where both the upper and lower bounds of this interval decrease as σ increases. This appears paradoxical, since at first glance it hence follows that even longer and more dangerous cracks should originate in solids, the smaller the loading on the body. However, it should not be forgotten that the greater the length of the equilibrium crack and the smaller the σ the greater will be the energy barrier which should be overcome for the appearance of a slot, and the smaller will be the probability of the origin of conditions at which this barrier (whose magnitude decreases as $1 / \sigma^4$ as σ increases since it is proportional to the crack area) will turn out to be surmountable. Thermal fluctuations, for example, will be one of the possible mechanisms of passage through the energy barrier. It is clear that the appearance of fluctuations in a unilateral direction in some volume of the body will be less and less probable the greater the volume within whose limits it will originate. Because of the above, it is evident that the formation of slots in elastic solids is possible only for sufficiently high values of σ i.e. for sufficiently small L_k.

As an illustration yielding a representation of the order of magnitude, let us present values of L_g, L_k and $2U_{\max}\ D^{-1}$ for $\beta = 1$ and several diverse values of σ/σ_c

$$\frac{2U_{\max}}{D} = 23, \qquad 9.30\cdot 10^3 D \leqslant L \leqslant 1.15\cdot 10^4 D \qquad \left(\frac{\sigma}{\sigma_c} = 0.01\right)$$

$$\frac{2U_{\max}}{D} = 2.8, \qquad 235\, D \leqslant L \leqslant 280\, D \qquad \left(\frac{\sigma}{\sigma_c} = 0.05\right)$$

$$\frac{2U_{\max}}{D} = 2.3, \qquad 93\, D \leqslant L \leqslant 115\, D \qquad \left(\frac{\sigma}{\sigma_c} = 0.1\right)$$

From these numbers it follows that the lengths of equilibrium cracks for stresses on the order of the yield point or strength of real solids turn out to be less than the grain size of polycrystals and more often commensurate with the size of blocks or spacings between dislocations. The maximum crack width hence varies between $3D$ and $35D$. The considered theory is thereby by no means a theory of microscopic cracks. It describes the mechanism of crack generation and propagation in polycristal grains. From it follows not only the possibility but also the naturalness of the formation of slots of microscopic and submicroscopic size in solids, even if the crystal lattices of these solids are defect-free. The presence of defects (vacancies, interstitial atoms, dislocations) exerts a double influence on propagation of slots of the kind considered. On one hand, the presence of defects can contribute to surmounting the energy barriers inhibiting crack formations, but on the other hand, defects and their clusters produce energy barriers in the path of the cracks being propagated. In particular, the grain boundaries and other clusters of retarded dislocations will be such barriers. Consequently, the regularities of microcrack propagation in real solids is more complex than those resulting from the examined theory, which solve the problem in an idealized formulation.

9. The perceptible value of the elucidated theory is in interpreting the brittle fracture mechanism as the loss of stability (in the large) of trivial equilibrium modes of the atomic lattices. The crack is hence considered not as a defect existing beforehand in the lattice (as has been assumed up to now), but as a nontrivial equilibrium mode of the deformation of an elastic solid, which becomes possible only in the presence of a loading stretching the solid. Thermal fluctuations and local imperfections, which always exist in atomic lattices, produce conditions under which energy barriers inhibiting the realization of such nontrivial defoemation modes are surmounted. In principle, equilibrium crack of a different kind are possible, including the infinitely long Griffith crack considered above, a circular crack, etc. However, the probabilities of their origination are not the same by far. Thus, for example, the energy barrier which must be surmounted for the formation of a Griffith crack, cutting through a solid, is very great (infinite if we speak of an infinitely large solid). Therefore, the origination of such cracks is not realistic, and should be considered as a classical example convenient for theoretical investigation, which permits a study of the qualitative aspect of the phenomenon. The concept proposed to distinguish cracks from the viewpoint of the probability of their formation discloses prospects of a statistical approach to the study of brittle fracture on the basis of an investigation of the physical conditions needed to surmount the appropriate energy barriers. The fact that the problem of equilibrium cracks, cuts, is nonlinear (according to the criterion of assigning boundary conditions on the contour whose size is not known in advance) has been voiced earlier.

However the physical nature of this nonlinearity has not been clear. On page 4 in [6] it is attributed to geometric factors. As has been shown above, the nonlinearity of atomic interaction actually plays the main role here, and the geometric nonlinearity of the problem can be neglected in a first approximation.

The second viewpoint in principle, propounded above, is the approach to brittle fracture as to a discrete process (the process of separation of atoms, in whose study the atomic diameter must be considered a finite quantity). Hence, a need arises to utilize the brittle strength criterion in the neighborhood of singular points of the stress field in the form (4.3) rather than the form (4.1). The latter is equivalent to rejecting the postulate of finiteness of the stress in the neighborhood of the crack ends. Hence, a finite range for the width (length) of a stable equilibrium crack of given specific form corresponds to each value of the tensile loading on the solid. When the postulate of finite stresses is included among the conditions of the problem, it is found that stable equilibrium cracks do not generally exist (if there are no external forces applied to the crack edges).

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